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Elliptic Curves Related to Cyclic Cubic Extensions II

by

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Abstract. We generalize some results in the earlier paper [4] and present further observations for elliptic curves arising from plane sections of a cubic surface defined by the norm map of an arbitrary cyclic cubic extension K/F.

Introduction

In the previous paper [4], we considered elliptic curves $\mathcal{X}_h$ defined over a number field F arising from plane sections of a cubic surface $\mathcal{X}$ defined by the norm map for a cyclic cubic extension K/F, and gave an upper bound for the Selmer groups of those elliptic curves using some ideal class group of the cyclic cubic extension K/F. Particularly we proved (Theorem 1.1, Remark 4.2 in [4]) that every elliptic curve $E := \mathcal{X}_h$ arises as above has the Weierstrass equation

$$y^2z + \mathrm{Tr}(\gamma_h)xyz + \mathrm{N}(\gamma_h)yz^2 = x^3 \quad (\gamma_h \in K^*)\,,$$

where Tr, N are the trace map and the norm map for K/F, respectively and the following two properties:

(i) E has an F-rational 3-torsion point T.
(ii) For the 3-isogeny $\phi : E \to \widehat{E} := E/\langle T\rangle$, there exists some point $P \in \widehat{E}(F)$ such that $K = F(\phi^{-1}(P))$.

In this paper, we generalize these results to any base field F of arbitrary characteristic (Theorem 1.1), and show that the converse statement holds; namely, if an elliptic curve E defined over F has the properties (i)(ii), then the elliptic curve E/F arises as some plane section of the cubic surface $\mathcal{X}$ associated with the cyclic cubic extension K/F. Especially the property (ii) connects with the rank of Mordell-Weil group; that is, (ii) is equivalent to the condition that the image of a certain connecting homomorphism δ is non-trivial; where δ is derived from the Galois cohomology induced by the 3-isogeny in (ii). It is well-known that if one knows the whole image of the map δ and its dual map $\widehat{\delta}$, then one can determine the rank rank$E(F)$ of the Mordell-Weil group $E(F)$. This result gives some sort of characterization of elliptic curves $\{E/F\}$ under the condition that the image $\mathrm{Im}\,\delta$ of δ is non-trivial (Theorem 1.2, Corollary 1.3). Finally, for every elliptic curve $\mathcal{X}_h$ over any field L containing F but not $\sqrt{-3}$, we prove that L-rational points on a certain Zariski open subset of $\mathcal{X}_h$ parametrize the set of all L-isomorphic elliptic curves $\{\mathcal{X}_{h'}\}$ $(\ni \mathcal{X}_h)$ passing through a fixed point (Theorem 2.27), which allows us to determine when two elliptic curves $\mathcal{X}_h$, $\mathcal{X}_{h'}$ are isomorphic over L (Theorem 1.4).

Our main results are stated in §1, and in §2 we give the proof of main results and present detailed arguments.

1. Setting and main results

Fix a cyclic cubic extension K/F over any field F, a generator σ of the Galois group $\mathrm{Gal}(K/F)$ (which we frequently identify with $\mathrm{Gal}(K(x, y, z, w)/F(x, y, z, w))$, where x, y, z, w are algebraically independent over F), and we write Tr, N for the trace map and the norm map for K/F (or $K(x, y, z, w)/F(x, y, z, w)$), respectively. For a basis $\{1, \alpha_1, \alpha_2\}$ of K/F, let $\mathcal{X}$ denote the cubic surface in $\mathbb{P}^3$ defined by

$$\mathrm{N}(x + y\alpha_1 + z\alpha_2) = w^3\,, \quad [x, y, z, w] \in \mathbb{P}^3\,,$$

which is independent of the choice of basis up to F-isomorphisms. In this paper, we call $\mathcal{X}$ the norm surface attached to K/F. We denote by $\mathcal{X}_h$ $(= \mathcal{X} \cap h)$ a plane section of $\mathcal{X}$ by a plane $h \subset \mathbb{P}^3$ defined over F, and let $\mathcal{E}(\mathcal{X})$ denote the set of all F-isomorphism classes of elliptic curves defined over F arising from plane sections $\{\mathcal{X}_h\}$ of $\mathcal{X}$ attached to K/F.

The following theorem generalizes Theorem 1.1 in [4] to arbitrary characteristic cases.

THEOREM 1.1. *Let $\mathcal{X}$ be a norm surface attached to any cyclic cubic extension K/F of arbitrary characteristic with any basis $\{1, \alpha_1, \alpha_2\}$. Then, for every F-isomorphism class $\{\mathcal{X}_h\} \in \mathcal{E}(\mathcal{X})$ of elliptic curves, there exists some representative $\mathcal{X}_h$ with a plane h of the form $a(x - w) + by + cz = 0$ and an isomorphism $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ over F. Here $\mathcal{W}(\mathcal{X}_h)$ is an elliptic curve defined over F given by the Weierstrass equation*

$$y^2z + \mathrm{Tr}(\gamma_h)xyz + \mathrm{N}(\gamma_h)yz^2 = x^3\,, \quad \text{where} \quad \gamma_h = \begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^\sigma & \alpha_2^\sigma \\ a & b & c \end{vmatrix}.$$

If an elliptic curve E defined over F has an isogeny ϕ defined over F, then let $(\widehat{E}, \widehat{\phi})$ denote the dual elliptic curve and isogeny for (E, ϕ).

THEOREM 1.2. *Let $\mathcal{E}_{K/F}$ be the set of all F-isomorphism classes of elliptic curves defined over F with the following two properties*:

(i) *E has an F-rational 3-torsion point T.*

(ii) *For the 3-isogeny $\phi : E \to \widehat{E} = E/\langle T\rangle$, there exists some point $P \in \widehat{E}(F)$ such that $K = F(\phi^{-1}(P))$.*

Then $\mathcal{E}_{K/F} = \mathcal{E}(\mathcal{X})$. Especially any elliptic curve satisfying the properties (i)(ii) *arises from the plane section of the norm surface $\mathcal{X}$ attached to K/F.*

Assume that F is a perfect field. Fix a separable closure $\overline{F}$ of F. For an elliptic curve E with an isogeny $\phi : E \to \widehat{E} = E/\langle T\rangle$ associated with some 3-torsion point T, if we let δ denote the connecting homomorphism

$$0 \longrightarrow \widehat{E}(F)/\phi(E(F)) \xrightarrow{\delta} H^1(F, \ker\phi) = \mathrm{Hom}(\mathrm{Gal}(\overline{F}/F), \mathbb{Z}/3\mathbb{Z})\,,$$

and $\widehat{\phi}$ the dual connecting homomorphism

$$0 \longrightarrow E(F)/\widehat{\phi}(\widehat{E}(F)) \xrightarrow{\widehat{\delta}} H^1(F,\ \ker\widehat{\phi})\,,$$

then there is a well-known formula (see [7], [5] for details)

$$\mathrm{rank}E(F) = \dim_{\mathbb{F}_3} \mathrm{Im}\,\delta + \dim_{\mathbb{F}_3} \mathrm{Im}\,\widehat{\delta} - \mu\,, \qquad \text{where} \qquad \mu = \begin{cases} 2 & \text{if } \sqrt{-3} \in F \\ 1 & \text{if } \sqrt{-3} \notin F\,. \end{cases}$$

Theorem 1.2 particularly says that if $\dim_{\mathbb{F}_3} \mathrm{Im}\,\delta > 0$ then the elliptic curve E arises as a plane section of a norm surface $\mathcal{X}$ attached to a certain cyclic cubic extension. Specifically, we have the following corollary.

COROLLARY 1.3. *Let E be an elliptic curve defined over a perfect field F with an F-rational 3-torsion point T, and ϕ the 3-isogeny $E \to \widehat{E} = E/\langle T\rangle$. If there exists some point $P \in \widehat{E}(F)$ not contained in $\phi(E(F))$ (i.e. $\mathrm{Im}\,\delta$ is non-trivial), then the elliptic curve E/F represents a class in $\mathcal{E}(\mathcal{X})$ for the norm surface $\mathcal{X}$ attached to the cyclic cubic extension $F(\phi^{-1}(P))/F$ (i.e. there exists some plane section $\mathcal{X}_h$ of $\mathcal{X}$ such that E is isomorphic to $\mathcal{X}_h$ over F).*

Further, the set of isomorphic elliptic curves $\{\mathcal{X}_h\}$ on the norm surface $\mathcal{X}$ turns out to be described by rational points on $\mathcal{X}$. In order to specify the statement, we recall the action of the algebraic torus $\mathcal{X}_{w\neq 0}$ on the dual projective space $\check{\mathbb{P}}^3$, the space of all planes $\{h\}$ in $\mathbb{P}^3$ (§2 in [4]). We frequently use the notation $V_{w\neq 0}$ for the Zariski open subset $V - \{w = 0\}$ of an algebraic variety $V \subset \mathbb{P}^3$.

Since the affine surface $\mathcal{X}_{w\neq 0}$ is a linear algebraic group defined over F, the left translation $Q \mapsto P \cdot Q$ on $\mathcal{X}_{w\neq 0}$ induces a faithful regular representation ρ and

$$\begin{array}{rccccc} \tilde{\rho} : \mathcal{X}_{w\neq 0}(\overline{F}) & \xrightarrow{\rho} & \mathrm{GL}_3(\overline{F}) & \longrightarrow & \mathrm{PGL}_4(\overline{F}) \\ P & \longmapsto & \rho(P) & \longmapsto & \begin{pmatrix} \rho(P) & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

The representation $\tilde{\rho}$ also induces an action of the group $\mathcal{X}_{w\neq 0}(\overline{F})$ on $\check{\mathbb{P}}^3$. Thus every point $P \in \mathcal{X}_{w\neq 0}(\overline{F})$ gives rise to the action

$$P \cdot \mathcal{X}_h := \tilde{\rho}(P)(\mathcal{X}_h) = \mathcal{X}_{\tilde{\rho}(P)(h)}\,.$$

Here the plane $\tilde{\rho}(P)(h) : a'x + b'y + c'z + d'w = 0$ is given by

$$[a',\ b',\ c',\ d'] = [a,\ b,\ c,\ d] \cdot {}^t\tilde{\rho}(P)^{-1}\,.$$

We identify a plane $ax + by + cz + dw = 0$ defined over a field L in $\mathbb{P}^3$ with a point $[a,\ b,\ c,\ d] \in \check{\mathbb{P}}^3(L)$, and denote the set of all planes defined over L in $\mathbb{P}^3$ by $\check{\mathbb{P}}^3(L)$.

The following theorem tells us when two elliptic curves $\mathcal{X}_h,\ \mathcal{X}_{h'}$ are isomorphic over F.

THEOREM 1.4. *Let $h,\ h' \in \check{\mathbb{P}}^3(F)$ be any planes such that $\mathcal{X}_h$ and $\mathcal{X}_{h'}$ are elliptic curves defined over F, and let L denote any field containing F but not $\sqrt{-3}$. Then $\mathcal{X}_h$ is isomorphic to $\mathcal{X}_{h'}$ over some field L if and only if there exists some point $P \in \mathcal{X}_{w\neq 0}(L)$ such that $\mathcal{X}_h = P \cdot \mathcal{X}_{h'}$.*

2. The proof of main results

Let K_t denote the minimal splitting field of the polynomial

$$g(x;t) := x^3 + tx^2 - (t+3)x + 1 \in F(t)[x]$$

with discriminant the square of $\delta(g) := t^2+3t+9$, which is a cyclic cubic extension over the function field $F(t)$. The polynomial $g(x;t)$ is generic for cyclic cubic extensions, namely, an appropriate specialization $t \in F$ gives a minimal polynomial $g(x;t)$ of an arbitrary cyclic cubic extension over F (See [6]). Note that, in characteristic 3, the polynomial $g(x;t)$ reduces to the the Artin-Schreier polynomial

$$x'^3 - x' - \frac{1}{t}$$

by the substitution $x' = 1/(x+1)$.

In the proofs of the theorems above, we will use the cyclic cubic extension $K_t/F(t)$ (which parametrizes all cyclic cubic extensions over F) instead of a single cyclic cubic extension K/F, and hence the setting for K/F in §1 is interpreted as setting for $K_t/F(t)$ in this section, such as $\mathrm{Gal}(K_t/F(t)) = \langle\sigma\rangle$. Note that the proof in this section is still valid for any specialization $t \in F$ such that $K_t \neq F$, but Theorem 2.5 is valid even for the case $K_t = F$ if $\delta(g) \neq 0$.

The proof of our results will be given in the following steps:

(I) $\mathcal{E}(\mathcal{X}) \subset \mathcal{E}_{K_t/F(t)}$.

(II) (Theorem 1.1) For every $F(t)$-isomorphism class $\{\mathcal{X}_h\} \in \mathcal{E}(\mathcal{X})$ of elliptic curves, there exists some representative $\mathcal{X}_h$ containing the point $\mathcal{O} := [1,\ 0,\ 0,\ 1] \in \mathcal{X}$ whose Weierstrass equation is given by

$$y^2z + \mathrm{Tr}(\gamma_h)xyz + \mathrm{N}(\gamma_h)yz^2 = x^3,$$

where γ_h is an element in K_t^* determined by h.

(III) $\mathcal{E}(\mathcal{X}) \supset \mathcal{E}_{K_t/F(t)}$.

(IV) Every elliptic curve $\mathcal{X}_h$ is isomorphic to a certain curve $\mathcal{C}_h$ in $\mathbb{P}^3$ over $F(t,\ h)$ defined using trace Tr and norm N.

(V) The rational points on the Zariski open subset $\mathcal{C}_{h,w\neq 0}$ on the curve $\mathcal{C}_h$ parametrize the isomorphic elliptic curves $\{\mathcal{X}_h\}$ passing through the point $\mathcal{O}$.

The polynomial $g(x;t)$ is used in the proof of (II), (III) and (IV). Although (II) was proved in [4] for the case where F is a number field, here we generalize it for an arbitrary field F and apply (II) to (III). In order to prove (I), we consider a certain divisor D on the norm surface $\mathcal{X}$, and show the existence of a special point on $\mathcal{X}_h$ derived from the divisor D. Theorem 1.2 follows from the results (I) and (III). Combining (IV) and (V) yields Theorem 1.4.

2.1. The proof of (I)

LEMMA 2.1. *For an elliptic curve E defined over $F(t)$, the properties* (i)(ii) *in Theorem 1.2 is equivalent to the following property*:

(iii) *There is a point $Q \in E(K_t)$ not contained in $E(F(t))$ such that* $\mathrm{Tr}(Q) = 3Q$. *Here* Tr *is the trace map from the Mordell-Weil group $E(K_t)$ to $E(F(t))$.*

Proof. Suppose that $E/F(t)$ has the properties (i)(ii). For any $Q \in \phi^{-1}(P)$, since $Q^{\sigma} \neq Q$ and $\phi(Q^{\sigma} - Q) = P^{\sigma} - P = O$ (identity element), we have $Q^{\sigma} - Q \in \ker\phi - \{O\}$. By the property (i), $\ker\phi = \langle T \rangle \subset E(F(t))$, and hence $(Q^{\sigma} - Q)^{\sigma^2} = Q^{\sigma} - Q$. Thus the property (iii) follows. Suppose that $E/F(t)$ has the property (iii). Then the point $T = Q^{\sigma} - Q$ is a 3-torsion point, and the condition $\mathrm{Tr}(Q) = 3Q$ is equivalent to $(Q^{\sigma} - Q)^{\sigma} = Q^{\sigma} - Q$, that is, $T^{\sigma} = T$. The property (i) follows from this. Let ϕ be an isogeny $E \to \widehat{E} = E/\langle T \rangle$ defined over $F(t)$. Then $O = \phi(T) = \phi(Q)^{\sigma} - \phi(Q)$, and the point $P = \phi(Q)$ satisfies the property (ii). □

LEMMA 2.2. *For any $a,\ b,\ c \in F(t)$, the determinant of the matrix*

$$M(a,\ b,\ c) = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^{\sigma} & \alpha_2^{\sigma} \\ a & b & c \end{pmatrix}$$

is zero if and only if $a = b = c = 0$.

Proof. If α is a root of the cubic equation $g(x; t) = 0$ then $\alpha,\ 1/(1-\alpha),\ 1 - 1/\alpha$ are all the roots. Let $\beta_1 := \alpha,\ \beta_2 := 1/(1-\alpha)$. Since $g(x; t)$ is a minimal polynomial of $K_t/F(t)$, $\{1,\ \beta_1,\ \beta_2\}$ is a basis of $K_t/F(t)$, and we can choose a root α so that $\beta_1^{\sigma} = \beta_2$. Then there exists a linear transformation $(1,\ \alpha_1,\ \alpha_2)M_0 = (1,\ \beta_1,\ \beta_2)$, where $M_0 \in \mathrm{GL}_2(F(t))$. We thus have

$$M(a,\ b,\ c) \cdot M_0 = \begin{pmatrix} 1 & \beta_1 & \beta_2 \\ 1 & \beta_1^{\sigma} & \beta_2^{\sigma} \\ a' & b' & c' \end{pmatrix} (=: M')$$

for some $a',\ b',\ c' \in F(t)$ (in fact, $a' = a$). Using the equation $g(\alpha; t) = 0$ yields

$$\beta_2 = -\alpha^2 - (t+1)\alpha + 2\,,$$
$$\beta_2^{\sigma} = 1 - \frac{1}{\alpha} = \alpha^2 + t\alpha - (t+2)\,.$$

Again using $g(\alpha; t) = 0$, the determinant $|M'|$ reduces to

$$\{a'(t-3)+b'(t+4)+2c'\}+\{a'(-t^2-t)+b'(-2t-1)+c'(-t-2)\}\alpha+(-a't-2b'-c')\alpha^2.$$

Since $\{1,\ \alpha,\ \alpha^2\}$ is also a basis of $K_t/F(t)$, the equation $|M'| = 0$ holds if and only if

$$\begin{pmatrix} t-3 & t+4 & 2 \\ -t^2-t & -2t-1 & -t-2 \\ -t & -2 & -1 \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a'(t-3) + b'(t+4) + 2c' \\ a'(-t^2-t) + b'(-2t-1) + c'(-t-2) \\ -a't - 2b' - c' \end{pmatrix}$$

is the zero vector. The coefficient matrix on the left-hand side has determinant $\delta(g)$, which is non-zero, and hence $|M(a,\ b,\ c)| \cdot |M_0| = |M'| = 0$ is equivalent to $(a,\ b,\ c)M_0 = (a',\ b',\ c') = (0,\ 0,\ 0)$. Now the lemma follows. □

The following proposition says that the plane section $D \cap h$ of a certain divisor D on $\mathcal{X}$ produces a special K_t-rational point on the elliptic curve $\mathcal{X}_h$.

PROPOSITION 2.3. *For the norm surface $\mathcal{X}$ attached to $K_t/F(t)$, let D be the divisor on $\mathcal{X}$ defined by the equation*

$$w = 0, \quad x + y\alpha_1 + z\alpha_2 = 0 .$$

For any plane $h \in \check{\mathbb{P}}^3(F(t))$, if $h \neq [0, 0, 0, 1]$ then the intersections $Q = D \cap h$, Q^σ, Q^{σ^2} are all the distinct points on $\mathcal{X}_h \cap \{w = 0\}$. Furthermore if we assume that the curve $\mathcal{X}_h$ is smooth (hence, is of genus 1) then the divisor $3(Q)$ is linearly equivalent to the divisor $(Q) + (Q^\sigma) + (Q^{\sigma^2})$ on $\mathcal{X}_h$.

Proof. For any $h = [a, b, c, d] \in \check{\mathbb{P}}^3(F(t)) - \{[0, 0, 0, 1]\}$, the intersection $Q = D \cap h$ is the point (which was denoted by $\mathcal{O}'$ in [4])

$$[c\alpha_1 - b\alpha_2,\ a\alpha_2 - c,\ -a\alpha_1 + b,\ 0] .$$

If $(D^\sigma \cap h =) Q^\sigma = Q$ then $D^\sigma \cap D \in h$, so

$$a(\alpha_1^\sigma\alpha_2 - \alpha_1\alpha_2^\sigma) + b(\alpha_2 - \alpha_2^\sigma) + c(\alpha_1^\sigma - \alpha_1) = 0 .$$

Here $D^\sigma \cap D = [\alpha_1^\sigma\alpha_2 - \alpha_1\alpha_2^\sigma,\ \alpha_2 - \alpha_2^\sigma,\ \alpha_1^\sigma - \alpha_1,\ 0]$. Since the left-hand side of this equation is just $|M(a, b, c)|$ defined in Lemma 2.2, we have $a = b = c = 0$. This contradicts $h \neq [0, 0, 0, 1]$. Therefore $Q^\sigma \neq Q$. Using the Galois action of $\mathrm{Gal}(K_t/F(t))$, it turns out that Q, Q^σ, Q^{σ^2} are distinct points, and clearly fill up $\mathcal{X}_h \cap \{w = 0\}$. Assume that $\mathcal{X}_h$ is smooth. Then the points Q, Q^σ, Q^{σ^2} are smooth, so the local ring $K_t[\mathcal{X}_h]_Q$ (resp. $K_t[\mathcal{X}_h]_{Q^\sigma}$, $K_t[\mathcal{X}_h]_{Q^{\sigma^2}}$) of $\mathcal{X}_h$ at Q (resp. Q^σ, Q^{σ^2}) is a discrete valuation ring. A direct calculation (similar calculation has been given in Lemma 6.1 in [4]) shows that the maximal ideal of the ring $K_t[\mathcal{X}_h]_Q$ (resp. $K_t[\mathcal{X}_h]_{Q^\sigma}$, $K_t[\mathcal{X}_h]_{Q^{\sigma^2}}$) is generated by the uniformizer w. Thus we have

$$\text{the principal divisor of } \frac{x + y\alpha_1 + z\alpha_2}{w} = 3(Q) - (Q) - (Q^\sigma) - (Q^{\sigma^2}) .$$

This proves the proposition. □

REMARK 2.4. Note that if $h = [0, 0, 0, 1]$ then the plane section $\mathcal{X}_h$ splits into three lines, namely, $\mathcal{X}_h = D \cup D^\sigma \cup D^{\sigma^2}$.

Now (I) follows from Proposition 2.3 and Lemma 2.1. If $\mathcal{X}_h$ is an elliptic curve representing an isomorphism class in $\mathcal{E}(\mathcal{X})$, then $\mathcal{X}_h$ is isomorphic to the degree 0 subgroup $\mathrm{Cl}^0(\mathcal{X}_h)$ of the Weil divisor class group of $\mathcal{X}_h$ over $F(t)$ via the Abel-Jacobi map, and hence, by Proposition 2.3 there exists a point $Q \in \mathcal{X}_h(K_t)$ satisfying the property (iii). Thus (I) follows.

2.2. The proof of (II) (Theorem 1.1)

For a plane $h \in \check{\mathbb{P}}^3(F(t))$, if there is some point $P \in \mathcal{X}_{h,w\neq 0}(F(t))$ then the point $\mathcal{O}$ lies on the plane $h' = \tilde{\rho}(P^{-1})(h)$ and $\mathcal{X}_{h'}$ is isomorphic to $\mathcal{X}_h$ over $F(t)$. It thus suffices to consider only the planes of the form

$$a(x - w) + by + cz = 0 , \quad [a, b, c, -a] \in \check{\mathbb{P}}^3(F(t))$$

on which the point $\mathcal{O}$ lies.

We first give a Weierstrass equation of an elliptic curve $\mathcal{X}_h$ for a special basis of $K_t/F(t)$.

THEOREM 2.5. *For any root α of the equation $g(x; t) = 0$, let $\mathcal{X}$ be a norm surface attached to $K_t/F(t)$ with the basis $\{1, \beta_1, \beta_2\}$, where $\beta_1 = \alpha$, $\beta_2 = 1/(1-\alpha)$. For any plane $h = [a, b, c, -a] \in \check{\mathbb{P}}^3(F(t))$ so that $\mathcal{X}_h$ is an elliptic curve, there exists an isomorphism $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ over $F(t)$. Here $\mathcal{W}(\mathcal{X}_h)$ is an elliptic curve defined in the statement of Theorem 1.1 (with the substitution $\{1, \alpha_1, \alpha_2\} \mapsto \{1, \beta_1, \beta_2\}$).*

REMARK 2.6. Theorem 2.5 is valid not only for any specialization $t \in F$ such that K_t/F is a cyclic cubic extension, but also for the case $K_t = F$ with a specialization $t \in F$ satisfying the condition $\delta(g) \neq 0$. In the latter case, we formally regard the set $\{1, \sigma, \sigma^2\}$ as the permutation group of order 3 acting on all the roots of $g(x; t) = 0$.

Upon using Theorem 2.5, we prove (II) (Theorem 1.1); that is, arbitrary basis cases.

Proof of Theorem 1.1. For any cyclic cubic extension K/F (which we identify with $K_t/F(t)$ for a suitable $t \in F$), since the set $\mathcal{X}_h(F)$ ($\neq \emptyset$) coincides with $\mathcal{X}_{h,w\neq 0}(F)$, we can choose a representative $\mathcal{X}_h$ passing through the point $\mathcal{O}$ from every F-isomorphism class $\{\mathcal{X}_h\} \in \mathcal{E}(\mathcal{X})$ as mentioned at the beginning of this subsection.

Let $\mathcal{X}'$ be the norm surface attached to K/F with the basis $\{1, \beta_1, \beta_2\}$, where $\beta_1 = \alpha$, $\beta_2 = 1/(1-\alpha)$ with a root α of the equation $g(x; t) = 0$. As we have seen in Lemma 2.2, there is a linear transformation $M_0 \in \mathrm{GL}_2(F)$ such that $M(a, b, c) \cdot M_0 = M'$, which induces an isomorphism $\mathcal{X}_h \simeq \mathcal{X}'_{h'}$ over F, where $h' = [a', b', c', -a']$. Thus, using Theorem 2.5 yields an isomorphism $\mathcal{X}'_{h'} \simeq \mathcal{W}(\mathcal{X}'_{h'})$ with $\gamma_{h'} = |M'| = |M(a, b, c)| \cdot |M_0|$. By the substitution $(x, y) \mapsto (|M_0|^2 x, |M_0|^3 y)$, we have $\mathcal{W}(\mathcal{X}'_{h'}) \simeq \mathcal{W}(\mathcal{X}_h)$, and hence $\mathcal{X}_h \simeq \mathcal{X}'_{h'} \simeq \mathcal{W}(\mathcal{X}'_{h'}) \simeq \mathcal{W}(\mathcal{X}_h)$. $\square$

REMARK 2.7. For the Weierstrass equation of $\mathcal{W}(\mathcal{X}_h)$, the discriminant is given by

$$\Delta(\mathcal{X}_h) = \mathrm{N}(\gamma_h)^3 \left\{ \mathrm{Tr}(\gamma_h)^3 - 27\mathrm{N}(\gamma_h) \right\}.$$

If $\Delta(\mathcal{X}_h)$ is non-zero (*i.e.* $\mathcal{X}_h$ is an elliptic curve), then the j-invariant is

$$j(\mathcal{X}_h) = \frac{\mathrm{Tr}(\gamma_h)^3 \left\{ \mathrm{Tr}(\gamma_h)^3 - 24\,\mathrm{N}(\gamma_h) \right\}^3}{\mathrm{N}(\gamma_h)^3 \left\{ \mathrm{Tr}(\gamma_h)^3 - 27\mathrm{N}(\gamma_h) \right\}}.$$

REMARK 2.8. In the paper [4], we saw in Theorem 1.1 that the plane section $\mathcal{X}_h$ has the Weierstrass equation

$$y^2 + a\,\delta(\mathcal{X})xy - \mathrm{N}(\gamma_h)y = x^3,$$

where $\delta(\mathcal{X})$ is the determinant

$$\delta(\mathcal{X}) = \begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^{\sigma} & \alpha_2^{\sigma} \\ 1 & \alpha_1^{\sigma^2} & \alpha_2^{\sigma^2} \end{vmatrix}.$$

However this statement is incorrect (The proof of Theorem 1.1 is correct). It should be replaced by

$$\delta(\mathcal{X}) = -\begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^{\sigma} & \alpha_2^{\sigma} \\ 1 & \alpha_1^{\sigma^2} & \alpha_2^{\sigma^2} \end{vmatrix}.$$

REMARK 2.9. It is easily seen that there is an equality

$$\mathrm{Tr}(\gamma_h) = a \begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^{\sigma} & \alpha_2^{\sigma} \\ 1 & \alpha_1^{\sigma^2} & \alpha_2^{\sigma^2} \end{vmatrix},$$

which gives an identification of the Weierstrass equation in Theorem 2.5 with that in Theorem 1.1 in [4] by the substitution $(x, y) \mapsto (x, -y)$.

Proof of Theorem 2.5. We first make preparations. If α is a root of the cubic equation $g(x; t) = 0$ then α, $1/(1-\alpha)$, $1 - 1/\alpha$ are all the roots. We can express the defining equation $f(x, y, z, w) = 0$ of $\mathcal{X}$ by

$$\begin{aligned} f(x, y, z, w) = x^3 - tx^2y - (t+3)xy^2 - y^3 \\ - tx^2z + (t^2+t+3)xyz - 3y^2z \\ - (t+3)xz^2 + (t^2+3t+6)yz^2 - z^3 - w^3 . \end{aligned}$$

Moreover, we make a substitution of the cubic curve $\mathcal{X}_h$ using the isomorphism

$$\begin{aligned} \vartheta_0 : \mathcal{X}_h &\longrightarrow \vartheta_0(\mathcal{X}_h) = \vartheta_0(\mathcal{X}) \cap \vartheta_0(h) \\ [x, y, z, w] &\longmapsto [x - w, y, z, w], \end{aligned}$$

which translates the point $\mathcal{O}$ to the point $P_1 := [0, 0, 0, 1]$. The surface $\vartheta_0(\mathcal{X})$ is defined by the equation $f_0(x, y, z, w) = 0$, where

$$f_0(x, y, z, w) = f(x + w, y, z, w),$$

and the plane $\vartheta_0(h)$ has the form

$$ax + by + cz = 0.$$

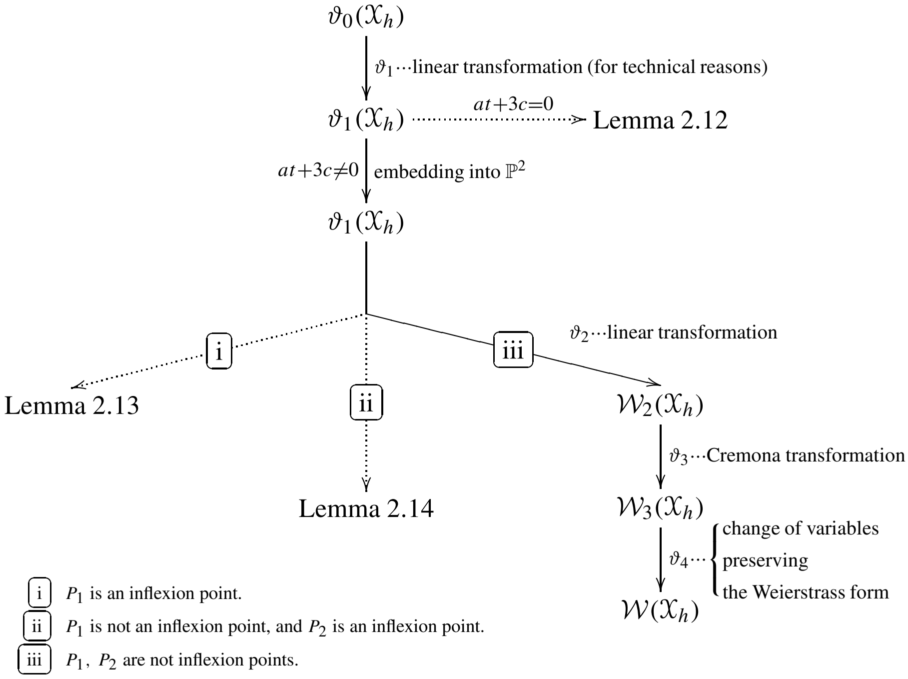
If P_1 is not an inflexion point then we denote by P_2 the third point at which the tangent line at P_1 meets $\vartheta_0(\mathcal{X}_h)$ again, and also if P_2 is not an inflexion point then let P_3 be the third point at which the tangent line at P_2 meets $\vartheta_0(\mathcal{X}_h)$ again.

On account of technical reasons, we divide the proof of Theorem 2.5 into two cases where $\mathrm{char}(F) \neq 3$ or $\mathrm{char}(F) = 3$. Each proof is given in Proposition 2.10 and Proposition 2.15, respectively. The proof we give here is based on the method in [1], [2]. $\square$

We assume the condition of Theorem 2.5 in the present subsection.

PROPOSITION 2.10. *If* $\mathrm{char}(F) \neq 3$ *then there exists an isomorphism* $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ *defined over* $F(t)$.

Proof. We will prove the statement via the following flowchart.



The first isomorphism ϑ_1 is defined by the linear map

$$\vartheta_1 : \vartheta_0(\mathcal{X}_h) \longrightarrow \vartheta_1(\mathcal{X}_h) = \vartheta_1(\mathcal{X}) \cap \vartheta_1(h)$$

$$[x,\ y,\ z,\ w] \longmapsto [x,\ y,\ z,\ w] \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 3 & 0 & 0 \\ t & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1},$$

which is well-defined by the assumption $\mathrm{char}(F) \neq 3$. This map is needed on account of technical reasons. The plane $\vartheta_1(h)$ has the form

$$ax + (at + 3b)y + (at + 3c)z = 0.$$

The case $at + 3c = 0$ will be treated in Lemma 2.12. Assume that $at + 3c \neq 0$. Then the above equation is equivalent to

$$Ax + By + z = 0, \quad \text{where} \quad A := \frac{a}{at + 3c}, \quad B := \frac{at + 3b}{at + 3c}.$$

After a calculation of the defining polynomial $f_0(\vartheta_1^{-1}([x,\ y,\ z,\ w]))$ of $\vartheta_1(\mathcal{X})$, making the substitution $z = -Ax - By$ yields the polynomial

$$f_1(x,\ y,\ w) := c_1x^3 + 3A\delta(g)c_2x^2y + 3\delta(g)c_3xy^2 + \delta(g)c_4y^3$$

$$- 3\{A^2\delta(g) - 1\}x^2w - 3A(2B+1)\delta(g)xyw - 3(B^2+B+1)\delta(g)y^2w + 3xw^2\,,$$

where

$$\begin{aligned}
c_1 &= (3A)^3 g\left(\frac{1-At}{3A};\, t\right) = A^3(2t+3)\delta(g) - 3A^2\delta(g) + 1\,,\\
c_2 &= A\{B(2t+3) + t + 6\} - 2B - 1\,,\\
c_3 &= A\{B^2(2t+3) + 2B(t+6) - t + 3\} - (B^2+B+1)\,,\\
c_4 &= (B+2)^3 g\left(\frac{-B+1}{B+2};\, t\right) = B^3(2t+3) + 3B^2(t+6) - 3B(t-3) - 2t - 3\,.
\end{aligned}$$

The equation $f_1(x,\, y,\, w) = 0$ defines the plane cubic curve $\vartheta_1(\mathcal{X}_h)$ in $\mathbb{P}^2$. It is verified that

$$\begin{aligned}
\vartheta_1(P_1) &= [0,\ 0,\ 1]\,,\\
\vartheta_1(P_2) &= \Big[0,\ 3(B^2+B+1),\ c_4\Big]\,,
\end{aligned}$$

and hence

$\boxed{\text{i}}$ if $B^2 + B + 1 = 0$ then the point $\vartheta_1(P_1)$ is an inflexion point.

This case will be treated in Lemma 2.13. We assume that $B^2 + B + 1 \neq 0$ (*i.e.* $\vartheta_1(P_1)$ is not an inflexion point). Then

$$\vartheta_1(P_3) = \Big[9(B^2+B+1)^3\delta(g)Sc_4,\ 3(B^2+B+1)(R+mS),\ Rc_4\Big]\,,$$

where

$$\begin{aligned}
m &= -9(B^2+B+1)^2\delta(g)c_3 + \{3A(B^3+(B+1)^3)\delta(g) - c_4\}c_4\,,\\
s &= -3A\{B^3+(B+1)^3\}\delta(g) + c_4\,,
\end{aligned}$$

$$\begin{aligned}
R = {}& 27(B^2+B+1)^6\delta(g)^2c_1 + 81(B^2+B+1)^4\delta(g)^2\{3A^2(B^2+B+1)\delta(g) - c_3\}c_3\\
&- 27A(B^2+B+1)^4\delta(g)^2c_2s - 18(B^2+B+1)^2\delta(g)c_3s^2 - c_4s^3\,,
\end{aligned}$$

$$\begin{aligned}
S = {}& 27A^2(B^2+B+1)^3\delta(g)^2\\
&- 9(B^2+B+1)^2\delta(g)\{B^2+B+1+3A^2B(B+1)\delta(g)+2c_3\}\\
&+ 9A(2B+1)(B^2+B+1)\delta(g)c_4 - 2c_4^2\,.
\end{aligned}$$

It thus turns out that

$\boxed{\text{ii}}$ if $Sc_4 = 0$ then the point $\vartheta_1(P_2)$ is an inflexion point.

$\boxed{\text{iii}}$ if $Sc_4 \neq 0$ then the point $\vartheta_1(P_2)$ is not an inflexion point.

The case $\boxed{\text{ii}}$ will be treated in Lemma 2.14. From now, we consider the case $\boxed{\text{iii}}$.

The second isomorphism ϑ_2 is defined by the linear map

$$\vartheta_2 : \vartheta_1(\mathcal{X}_h) \longrightarrow \mathcal{W}_2(\mathcal{X}_h)$$

$$[x,\ y,\ w] \longmapsto [x,\ y,\ w] \cdot \begin{pmatrix} x(\vartheta_1(P_1)) & y(\vartheta_1(P_1)) & w(\vartheta_1(P_1)) \\ x(\vartheta_1(P_2)) & y(\vartheta_1(P_2)) & w(\vartheta_1(P_2)) \\ x(\vartheta_1(P_3)) & y(\vartheta_1(P_3)) & w(\vartheta_1(P_3)) \end{pmatrix}^{-1},$$

which satisfies

$$\vartheta_2(\vartheta_1(P_1)) = [1,\ 0,\ 0], \quad \vartheta_2(\vartheta_1(P_2)) = [0,\ 1,\ 0], \quad \vartheta_2(\vartheta_1(P_3)) = [0,\ 0,\ 1]\,.$$

Since the points $\vartheta_1(P_1),\ \vartheta_1(P_2),\ \vartheta_1(P_3)$ are not collinear, the map ϑ_2 is a well-defined isomorphism. Then the equation $f_2(x,\ y,\ w) = 0$ defining $\mathcal{W}_2(\mathcal{X}_h)$ is given by

$$\begin{aligned} f_2(x, y, w) &:= -\frac{1}{\delta(g)} f_1\left(3(B^2 + B + 1)^2\delta(g)Sc_4w,\ y+(R+mS)w,\ \frac{x + c_4y + Rc_4w}{3(B^2 + B + 1)}\right) \\ &= xy^2 - Sc_4x^2w - Q_1xyw - Q_2xw^2 + S^3c_4^3yw^2\,, \end{aligned}$$

where

$$\begin{aligned} Q_1 &= 9A(B^2 + B + 1)\delta(g)Sc_4 + 2q\,, \\ Q_2 &= -3p + q(q - Q_1)\,, \\ p &= \frac{1}{3}S^2c_4^2(S - c_4^2)\,, \\ q &= \{-3A(B + 2)(B^2 + B + 1)\delta(g) + c_4\}Sc_4 - (R + mS)\,. \end{aligned}$$

The third isomorphism ϑ_3 is the Cremona transformation (§4.4 in [2])

$$\begin{aligned} \vartheta_3 : \mathcal{W}_2(\mathcal{X}_h) &\longrightarrow \mathcal{W}_3(\mathcal{X}_h) \\ [x,\ y,\ w] &\longmapsto [xw,\ xy,\ w^2]\,, \end{aligned}$$

whose inverse morphism is given by $[x,\ y,\ w] \mapsto [x^2,\ yw,\ xw]$. Since the curves $\mathcal{W}_2(\mathcal{X}_h),\ \mathcal{W}_3(\mathcal{X}_h)$ are smooth by assumption, the birational map ϑ_3 is an isomorphism. We thus find the defining polynomial

$$\frac{1}{x^2w} f_2(x^2,\ yw,\ xw) = y^2w - Q_1xyw + S^3c_4^3yw^2 - Sc_4x^3 - Q_2x^2w$$

of $\mathcal{W}_3(\mathcal{X}_h)$. Scaling by $w \mapsto w/(Sc_4)$ yields the polynomial of the Weierstrass form

$$\begin{aligned} f_3(x,\ y,\ w) &:= \frac{1}{Sc_4x^2w} f_2(x^2,\ Sc_4yw,\ Sc_4xw) \\ &= y^2w - Q_1xyw + S^4c_4^4yw^2 - x^3 - Q_2x^2w\,. \end{aligned}$$

The final isomorphism ϑ_4 is the following change of variables preserving the Weierstrass form.

$$\begin{aligned} \vartheta_4 : \mathcal{W}_3(\mathcal{X}_h) &\longrightarrow \mathcal{W}(\mathcal{X}_h) \\ [x,\ y,\ w] &\longmapsto \left[\frac{1}{u^2}(x - pw),\ \frac{1}{u^3}\{y - qx + (pq - r)w\},\ w\right], \end{aligned}$$

where

$$u = \frac{3}{at + 3c} 3(B^2 + B + 1)Sc_4,$$

$$\begin{aligned} r = -\{&3A(B^2+B+1)\delta(g) - B^3(t-3) - 3B^2(2t+3) - 3B(t+6) + t - 3\}^3 \\ &\times \{(B^3(t+6) - 3B^2(t-3) - 3B(2t+3) - t - 6)S \\ &\quad + 9(A(B^2+B+1)\delta(g) + B^3 - B^2 t - B(t+3) - 1)c_4^2\}S^2 c_4^2 . \end{aligned}$$

Then the defining polynomial of $\mathcal{W}(\mathcal{X}_h)$ is

$$\frac{1}{u^6} f_3(u^2 x + pz,\ u^3 y + u^2 q x + rz,\ z) = y^2 z + \mathrm{Tr}(\gamma_h) xyz + \mathrm{N}(\gamma_h) yz^2 - x^3 .$$

Here

$$\begin{aligned} \mathrm{Tr}(\gamma_h) &= -a\delta(g), \\ \mathrm{N}(\gamma_h) &= \delta(g)\{b^3 - c^3 - b^2 ct - bc^2(t+3) - a^3(2t+3) + a^2(b(t+3) + 3c)t \\ &\qquad + a(b^2(2t+3) - bc(t^2+t-3) - c^2(t-3))\} \end{aligned}$$

for the basis $\{1,\ \beta_1,\ \beta_2\}$ of $K_t/F(t)$. This completes the proof. □

REMARK 2.11. From the proof of Proposition 2.10, we have an isomorphism

$$\begin{aligned} \mathcal{W}_3(\mathcal{X}_h) &\longrightarrow \vartheta_1(\mathcal{X}_h) \\ [x,\ y,\ w] &\longmapsto [x',\ y',\ w'] , \end{aligned}$$

where

$$\begin{aligned} x' &= 3(B^2+B+1)^2\delta(g) S^2 c_4^2 xw , \\ y' &= (R + mS)Sc_4 xw + Sc_4 yw , \\ w' &= \frac{x^2 + RSc_4^2 xw + Sc_4^2 yw}{3(B^2+B+1)} . \end{aligned}$$

LEMMA 2.12. *If* $\mathrm{char}\,(F) \neq 3$ *and* $at + 3c = 0$ *then there exists an isomorphism* $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ *defined over* $F(t)$.

Proof. Since the proof is based on a standard method (see §4.4 in [2] for details), we omit the detailed argument here. Let ϑ_1 be the isomorphism defined in the proof of Proposition 2.10. If $at + 3b = 0$ then the plane $\vartheta_1(h) : x = 0$ is tangent to the norm surface $\mathcal{X}$, and hence the plane section $\mathcal{X}_h$ is singular. We thus assume $at + 3b \neq 0$. Then

$$\vartheta_1(h) : Ax + y = 0 , \quad \text{where} \quad A := \frac{a}{at+3b} .$$

The defining polynomial of $\vartheta_1(\mathcal{X}_h)$ in $\mathbb{P}^3$ is

$$\begin{aligned} f_1(x,\ z,\ w) &:= f_0(\vartheta_1^{-1}([x,\ -Ax,\ z,\ w])) \\ &= \{A^3(2t+3)\delta(g) - 3A^2\delta(g) + 1\}x^3 + 3A\{A(t-3) - 1\}\delta(g)x^2 z \\ &\quad - 3\{A(t+6) + 1\}\delta(g)xz^2 - (2t+3)\delta(g)z^3 \\ &\quad - 3\{A^2\delta(g) - 1\}x^2 w - 3A\delta(g)xzw - 3\delta(g)z^2 w + 3xw^2 . \end{aligned}$$

Let

$$R = 729A^3\delta(g)^3 + 81A^2\delta(g)^2\{2\delta(g) + 27\} + 81A\{5\delta(g) + 27\}\delta(g)$$

$$
\begin{aligned}
&\quad - 2\{\delta(g) - 135\}\delta(g) + 729\,, \\
S &= -27A^2\delta(g)^2 - 81A\delta(g) - \delta(g) - 54\,, \\
m &= -\{3A(t+15) + 5\}\delta(g) - 27\,.
\end{aligned}
$$

It turns out that

$$
\begin{aligned}
\vartheta_1(P_1) &= [0,\ 0,\ 1]\,, \\
\vartheta_1(P_2) &= [0,\ 3,\ -(2t+3)]\ , \\
\vartheta_1(P_3) &= [9(2t+3)\delta(g)S,\ 3R,\ -(2t+3)(R - mS)]\ .
\end{aligned}
$$

There are isomorphisms

$$
\mathcal{W}(\mathcal{X}_h) \longrightarrow \mathcal{W}_4(\mathcal{X}_h) \longrightarrow \vartheta_1(\mathcal{X}_h)
$$

defined as follows.

$\boxed{\mathrm{i}}$ the case where $(2t+3)S = 0$: (*i.e.* $\vartheta_1(P_2)$ is an inflexion point.)
The defining polynomial of $\mathcal{W}_4(\mathcal{X}_h)$ is

$$
f_4(x,\ z,\ w) := \frac{1}{27\delta(g)k} f_1(x',\ z',\ w')\,,
$$

where

$$
\begin{aligned}
x' &= -9\delta(g)x\,, \\
z' &= 3z\,, \\
w' &= -mx - (2t+3)z - kw\,, \\
k &= \{27A\delta(g) + 2t^2 + 6t + 45\}(2t+3)^2 - \{3A\delta(g) - t + 3\}(t+6)S\,.
\end{aligned}
$$

If $2t + 3 = 0$ then $S \neq 0$ by the irreducibility of $\mathcal{X}_h$, and this condition implies $k \neq 0$. Then we have an isomorphism

$$
\begin{aligned}
\mathcal{W}_4(\mathcal{X}_h) &\longrightarrow \vartheta_1(\mathcal{X}_h) \\
[x,\ z,\ w] &\longmapsto [x',\ z',\ w']\,.
\end{aligned}
$$

There is a change of variables preserving the Weierstrass form defined by

$$
\begin{aligned}
\mathcal{W}(\mathcal{X}_h) &\longrightarrow \mathcal{W}_4(\mathcal{X}_h) \\
[x,\ y,\ z] &\longmapsto [u^2x + pz,\ u^3y + u^2qx + rz,\ z]\,,
\end{aligned}
$$

where

$$
\begin{aligned}
u &= \frac{9}{at + 3b}\,, \\
p &= \{3A\delta(g) - t + 3\}\{3A\delta(g) + t + 6\}\,, \\
q &= -3A\delta(g) + 2t + 3\,, \\
r &= \begin{cases} \frac{1}{3}(2t+3)^2\{9A\delta(g) - 2t + 6\} - \frac{1}{6}(4t+15)S & \text{if } \operatorname{char}(F) \neq 2\,, \\ \frac{1}{3}(2t+3)^2\{9A\delta(g) - 2t + 6\} & \text{if } \operatorname{char}(F) = 2\,. \end{cases}
\end{aligned}
$$

$\boxed{\text{ii}}$ the case where $(2t+3)S \neq 0$: (*i.e.* $\vartheta_1(P_2)$ is not an inflexion point.)
The defining polynomial of $\mathcal{W}_4(\mathcal{X}_h)$ is

$$f_4(x,\ z,\ w) := \frac{1}{27(2t+3)^2\delta(g)S^2x^2w} f_1(x',\ z',\ w')\,.$$

Then we have an isomorphism

$$\begin{aligned} \mathcal{W}_4(\mathcal{X}_h) &\longrightarrow \vartheta_1(\mathcal{X}_h) \\ [x,\ z,\ w] &\longmapsto [x',\ z',\ w']\,, \end{aligned}$$

where

$$\begin{aligned} x' &= 9(2t+3)^2\delta(g)S^2xw\,, \\ z' &= -3(2t+3)(R-mS)Sxw - 3(2t+3)Szw\,, \\ w' &= x^2 + (2t+3)^2RSxw + (2t+3)^2Szw\,. \end{aligned}$$

There is a change of variables preserving the Weierstrass form defined by

$$\begin{aligned} \mathcal{W}(\mathcal{X}_h) &\longrightarrow \mathcal{W}_4(\mathcal{X}_h) \\ [x,\ y,\ z] &\longmapsto [u^2x + pz,\ u^3y + u^2qx + rz,\ z]\,, \end{aligned}$$

where

$$\begin{aligned} u &= \frac{9(2t+3)S}{at+3b}\,, \\ p &= (2t+3)^2\{9A^2\delta(g)^2 + (27A-1)\delta(g) + 27\}S^2\,, \\ q &= 3\{3A\delta(g) - t + 3\}^2\{9A(t+3)\delta(g) + \delta(g) + 9(t+3)\}\,, \\ r &= (2t+3)^2\{3A\delta(g) - t + 3\}^3\{27A^2(t+6)\delta(g)^2 \\ &\qquad + 9A\delta(g)(4\delta(g) + 9(t+3)) + (t+42)\delta(g) + 54t + 81\}S^2\,. \end{aligned}$$

□

LEMMA 2.13. *With notation as in the proof of Proposition 2.10, if* char$(F) \neq 3$, $at + 3c \neq 0$, *and the point* $\vartheta_1(P_1) \in \vartheta_1(\mathcal{X}_h)$ *is an inflexion point then there exists an isomorphism* $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ *defined over* $F(t)$.

Proof. The condition that the point $\vartheta_1(P_1)$ is an inflexion point implies $B^2+B+1 = 0$, and $c_4 \neq 0$ by the irreducibility of $\mathcal{X}_h$. Here we give only isomorphisms

$$\mathcal{W}(\mathcal{X}_h) \longrightarrow \mathcal{W}_5(\mathcal{X}_h) \longrightarrow \vartheta_1(\mathcal{X}_h)\,.$$

The defining polynomial of $\mathcal{W}_5(\mathcal{X}_h)$ is

$$f_5(x,\ y,\ z) := \frac{1}{\delta(g)c_4} f_1\left(\frac{\delta(g)c_4}{3}z,\ -x,\ y\right)\,.$$

Then we have an isomorphism

$$\begin{aligned} \mathcal{W}_5(\mathcal{X}_h) &\longrightarrow \vartheta_1(\mathcal{X}_h) \\ [x,\ y,\ z] &\longmapsto \left[\frac{\delta(g)c_4}{3}z,\ -x,\ y\right]\,. \end{aligned}$$

There is a change of variables preserving the Weierstrass form defined by

$$\begin{aligned}\mathcal{W}(\mathcal{X}_h) &\longrightarrow \mathcal{W}_5(\mathcal{X}_h)\\ [x,\ y,\ z] &\longmapsto [u^2x + pz,\ u^3y + u^2qx + rz,\ z]\,,\end{aligned}$$

where

$$\begin{aligned}u &= \frac{3}{at + 3c}\,,\\ p &= A\delta(g)\{A\delta(g) + 3B - t\}\,,\\ q &= -A(B + 2)\delta(g)\,,\\ r &= \delta(g)\{-A^3(B + 1)\delta(g)^2 + (B + 1)t + 3\}\,.\end{aligned}$$

□

LEMMA 2.14. *With notation as in the proof of Proposition 2.10, if* char$(F) \neq 3$, $at + 3c \neq 0$, *and the point* $\vartheta_1(P_2) \in \vartheta_1(\mathcal{X}_h)$ *is an inflexion point then there exists an isomorphism* $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ *defined over* $F(t)$.

Proof. The condition that the point $\vartheta_1(P_2)$ is an inflexion point implies $Sc_4 = 0$. Here we give only isomorphisms

$$\mathcal{W}(\mathcal{X}_h) \longrightarrow \mathcal{W}_6(\mathcal{X}_h) \longrightarrow \vartheta_1(\mathcal{X}_h)\,.$$

The defining polynomial of $\mathcal{W}_6(\mathcal{X}_h)$ is

$$f_6(x,\ y,\ w) := -\frac{1}{27(B^2 + B + 1)^9\delta(g)k} f_1(x',\ y',\ w')\,,$$

where

$$\begin{aligned}x' &= 9(B^2 + B + 1)^5\delta(g)x\,,\\ y' &= 3(B^2 + B + 1)^4y\,,\\ w' &= -(B^2 + B + 1)^2mx + (B^2 + B + 1)^3c_4y + kw\,,\\ k &= \{9(B^2 + B + 1)^3(A^2\delta(g) - 1)\delta(g) + m\}m + 27(B^2 + B + 1)^6\delta(g)^2c_1\,.\end{aligned}$$

If $c_4 = 0$ then $S \neq 0$ by the irreducibility of $\mathcal{X}_h$, and this condition implies $k \neq 0$. Then we have an isomorphism

$$\begin{aligned}\mathcal{W}_6(\mathcal{X}_h) &\longrightarrow \vartheta_1(\mathcal{X}_h)\\ [x,\ y,\ w] &\longmapsto [x',\ y',\ w']\,.\end{aligned}$$

There is a change of variables preserving the Weierstrass form defined by

$$\begin{aligned}\mathcal{W}(\mathcal{X}_h) &\longrightarrow \mathcal{W}_6(\mathcal{X}_h)\\ [x,\ y,\ z] &\longmapsto [u^2x + pz,\ u^3y + u^2qx + rz,\ z]\,,\end{aligned}$$

where

$$u = \frac{9}{at + 3c}\,,$$

$$p = \frac{S - c_4^2}{3(B^2 + B + 1)^2}\,,$$

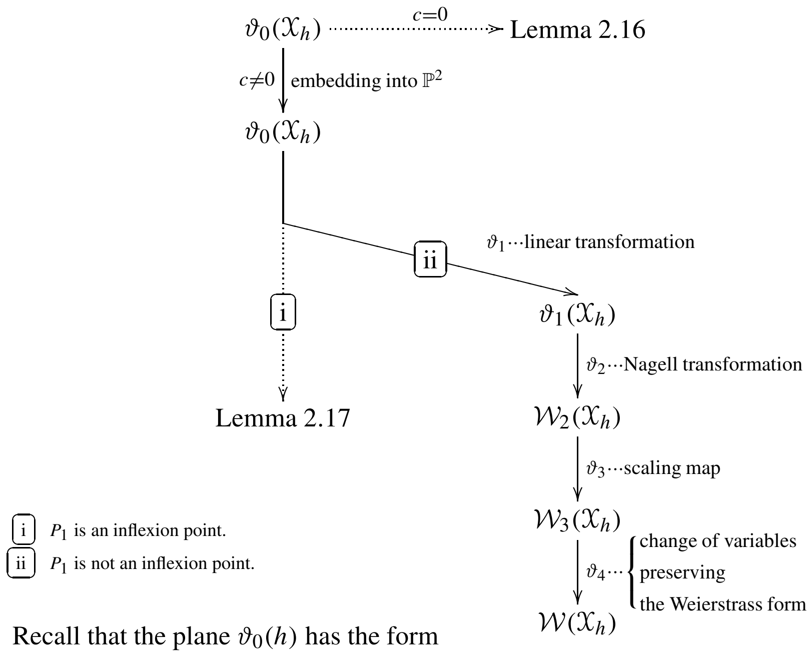
$$q = \frac{-3A\{(B+1)^3 + 1\}\delta(g) + c_4}{B^2 + B + 1}\,,$$

$$r = \begin{cases} \frac{1}{3}(2t+3)^2\{9A\delta(g) - 2t + 6\} - \frac{1}{6}(4t+15)S & \text{if char}\,(F) \neq 2\,, \\ \frac{1}{3}(2t+3)^2\{9A\delta(g) - 2t + 6\} & \text{if char}\,(F) = 2\,. \end{cases}$$

□

PROPOSITION 2.15. *If* char$\,(F) = 3$ *then there exists an isomorphism* $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ *defined over* $F(t)$.

Proof. We will prove the statement via the following flowchart.



Recall that the plane $\vartheta_0(h)$ has the form

$$ax + by + cz = 0\,.$$

The case $c = 0$ will be treated in Lemma 2.16. Assume that $c \neq 0$. Then the above equation is equivalent to

$$Ax + By + z = 0\,, \quad \text{where} \quad A := \frac{a}{c}\,, \quad B := \frac{b}{c}\,.$$

For the defining polynomial $f_0(x,\ y,\ z,\ w)$ of $\vartheta_0(\mathcal{X})$, making the substitution $z = -Ax - By$ yields the polynomial

$$\begin{aligned} f_0(x,\ y,\ w) := {} & (A^3 - A^2t + At + 1)x^3 + \{A^2t^2 - A(-B+t+1)t + (B-1)t\}x^2y \\ & - \{ABt + B^2 + B(t+1) + 1\}txy^2 + (B^3 + B^2t^2 - 1)y^3 \end{aligned}$$

$$- (A+1)Atx^2w - \{A(-B+t+1)+B-1\}txyw$$
$$- \{B^2+B(t+1)+1\}ty^2w + Atxw^2 + (B-1)tyw^2 \,.$$

The equation $f_0(x,\ y,\ w) = 0$ defines the plane cubic curve $\vartheta_0(\mathcal{X}_h)$ in $\mathbb{P}^2$. It is verified that

$$P_1 = [0,\ 0,\ 1]\,,$$
$$P_2 = \left[-A^2(B-1)t^3,\ A^3t^3,\ (B-1)^3 g\left(\frac{At-B+1}{B-1};\ t\right)\right] \quad (=: [x_0,\ y_0,\ w_0])\,,$$

and hence

$\boxed{\text{i}}$ if $A = 0$ then the point P_1 is an inflexion point.

$\boxed{\text{ii}}$ if $A \neq 0$ then the point P_1 is not an inflexion point.

The case $\boxed{\text{i}}$ will be treated in Lemma 2.17. From now, we consider the case $\boxed{\text{ii}}$.

The first isomorphism ϑ_1 is defined by the linear map

$$\vartheta_1 : \vartheta_0(\mathcal{X}_h) \longrightarrow \vartheta_1(\mathcal{X}_h)$$
$$[x,\ y,\ w] \longmapsto [x,\ y,\ w] \cdot \begin{pmatrix} y_0 & 0 & 0 \\ -x_0 & -w_0 & 1-w_0 \\ 0 & y_0 & y_0 \end{pmatrix},$$

which translates the point P_1, P_2 to the points [0, 1, 1], [0, 0, 1], respectively. Thus, we can reduce the argument to §8 in [1]. The curve $\vartheta_1(\mathcal{X}_h)$ has the defining equation $f_1(x,\ y,\ w) = 0$, where

$$f_1(x,\ y,\ w) := f_0(x - x_0y + x_0w,\ y_0(-y+w),\ (1-w_0)y + w_0w)\ .$$

Let

$$f_1(x,\ y,\ 1) = F_3(x,\ y) + F_2(x,\ y) + F_1(x,\ y)\,,$$

where F_i is a homogeneous polynomial of degree i.

The second isomorphism ϑ_2 is given by the Nagell transformation

$$\vartheta_2 : \vartheta_1(\mathcal{X}_h) \longrightarrow \mathcal{W}_2(\mathcal{X}_h)$$
$$[x,\ y,\ w] \longmapsto \left[\frac{y}{x}w,\ 2F_3\left(1,\ \frac{y}{x}\right)x + F_2\left(1,\ \frac{y}{x}\right)w,\ w\right].$$

The equation $f_2(x,\ y,\ w) = 0$ defining $\mathcal{W}_2(\mathcal{X}_h)$ is given by

$$f_2(x, y, w) := y^2w - A^7t^9x^3 - A^4t^6q^2x^2w + At^3q(pq-r)xw^2 + \frac{1}{A^2}\{p^3 - (pq-r)^2\}w^3\,,$$

where

$$p = At\{A^3t(t+1) - A^2(B+t)t + A(B^2+B(t+1)+1) + B^2+B+1\}\,,$$
$$q = A^2t^2\,,$$
$$r = A^3t^2\{-A^2(A+Bt)t + A(B^2+B(t+1)+1)t - B^3 + B(B+1)t + 1\}\,.$$

The third isomorphism ϑ_3 is defined by the scaling map

$$\vartheta_3 : \mathcal{W}_2(\mathcal{X}_h) \longrightarrow \mathcal{W}_3(\mathcal{X}_h)$$

$$[x,\ y,\ w] \longmapsto \left[A^2t^3x,\ y,\ \frac{w}{A}\right].$$

The equation $f_3(x,\ y,\ w) = 0$ defining $\mathcal{W}_3(\mathcal{X}_h)$ is given by

$$\begin{aligned} f_3(x,\ y,\ w) &:= \frac{1}{A} f_2\Big(\frac{x}{A^2t^3},\ y,\ Aw\Big) \\ &= y^2w - x^3 - q^2x^2w - q(pq-r)xw^2 + \{p^3 - (pq-r)^2\}w^3 . \end{aligned}$$

The final isomorphism ϑ_4 is the following change of variables preserving the Weierstrass form.

$$\begin{aligned} \vartheta_4 : \mathcal{W}_3(\mathcal{X}_h) &\longrightarrow \mathcal{W}(\mathcal{X}_h) \\ [x,\ y,\ w] &\longmapsto \left[\frac{1}{u^2}(x - pw),\ \frac{1}{u^3}\{y - qx + (pq - r)w\},\ w\right], \end{aligned}$$

where $u = A/c$. Then the defining polynomial of $\mathcal{W}(\mathcal{X}_h)$ is

$$\frac{1}{u^6} f_3(u^2x + pz,\ u^3y + u^2qx + rz,\ z) = y^2z + \mathrm{Tr}(\gamma_h)xyz + \mathrm{N}(\gamma_h)yz^2 - x^3 .$$

Here

$$\begin{aligned} \mathrm{Tr}(\gamma_h) &= -a\delta(g), \\ \mathrm{N}(\gamma_h) &= \delta(g)\{b^3 - c^3 - b^2ct - bc^2(t+3) - a^3(2t+3) + a^2(b(t+3) + 3c)t \\ &\qquad + a(b^2(2t+3) - bc(t^2+t-3) - c^2(t-3))\} \end{aligned}$$

for the basis $\{1,\ \beta_1,\ \beta_2\}$ of $K_t/F(t)$. This completes the proof. □

LEMMA 2.16. *If* $\mathrm{char}\,(F) = 3$ *and* $c = 0$ *then there exists an isomorphism* $\vartheta : \mathcal{X}_h \simeq \mathcal{W}(\mathcal{X}_h)$ *defined over* $F(t)$.

Proof. Since the proof is based on a standard method (see §8 in [1] for details), we omit the detailed argument here. If $b = 0$ then the plane $\vartheta_0(h) : x = 0$ is tangent to the norm surface $\mathcal{X}$, and hence the plane section $\mathcal{X}_h$ is singular. We thus assume $b \neq 0$. Then

$$\vartheta_0(h) : Ax + y = 0, \quad \text{where} \quad A := \frac{a}{b}.$$

The defining polynomial of $\vartheta_0(\mathcal{X}_h)$ in $\mathbb{P}^3$ is

$$\begin{aligned} f_0(x,\ z,\ w) &:= f_0(x,\ -Ax,\ z,\ w) \\ &= (A^3 - A^2t + At + 1)x^3 - \{A(t+1) + 1\}tx^2z - (At+1)txz^2 - z^3 \\ &\qquad - A(A+1)tx^2w - \{A(t+1) - 1\}txzw - tz^2w + Atxw^2 - tzw^2. \end{aligned}$$

It turns out that

$$\begin{aligned} P_1 &= [0,\ 0,\ 1], \\ P_2 &= \big[-A^2t^3,\ -A^3t^3,\ g(At-1;t)\big] \quad (=: [x_0,\ z_0,\ w_0]). \end{aligned}$$

$\boxed{\text{i}}$ the case where $A = 0$: (*i.e.* P_1 is an inflexion point.)
There is an isomorphism

$$\begin{aligned} \vartheta_0(\mathcal{X}_h) &\longrightarrow \mathcal{W}(\mathcal{X}_h) \\ [x,\ z,\ w] &\longmapsto [t(x-z),\ t^2(x+w),\ z]\,. \end{aligned}$$

$\boxed{\text{ii}}$ the case where $A \neq 0$: (*i.e.* P_1 is not an inflexion point.)
There are isomorphisms

$$\vartheta_0(\mathcal{X}_h) \longrightarrow \vartheta_1(\mathcal{X}_h) \longrightarrow \mathcal{W}_4(\mathcal{X}_h) \longrightarrow \mathcal{W}_5(\mathcal{X}_h) \longrightarrow \mathcal{W}(\mathcal{X}_h)$$

defined as follows.

The first isomorphism is defined by the linear map

$$\begin{aligned} \vartheta_0(\mathcal{X}_h) &\longrightarrow \vartheta_1(\mathcal{X}_h) \\ [x,\ z,\ w] &\longmapsto [x,\ z,\ w] \cdot \begin{pmatrix} z_0 & -w_0 & 1-w_0 \\ -x_0 & 0 & 0 \\ 0 & x_0 & x_0 \end{pmatrix}\,, \end{aligned}$$

where the defining polynomial of $\vartheta_1(\mathcal{X}_h)$ is

$$f_1(x,\ z,\ w) := f_0(x_0(-z+w),\ -x - z_0 z + z_0 w,\ (1-w_0)z + w_0 w)\ .$$

Let

$$f_1(x,\ z,\ 1) = F_3(x,\ z) + F_2(x,\ z) + F_1(x,\ z)\,,$$

where F_i is a homogeneous polynomial of degree i.

The defining polynomial of $\mathcal{W}_4(\mathcal{X}_h)$ is

$$f_4(x,\ y,\ w) := y^2 w - A^6 t^9 x^3 - A^4 t^6 q^2 x^2 w - A^2 t^3 q(pq-r)xw^2 + \{p^3 - (pq-r)^2\}w^3\,,$$

where

$$\begin{aligned} p &= \{A(At + A - 1)t + 1\}t\,, \\ q &= At^2\,, \\ r &= -\{A(A^2 + At - 1)t + 1\}t^2\,. \end{aligned}$$

Then we have an isomorphism

$$\begin{aligned} \vartheta_1(\mathcal{X}_h) &\longrightarrow \mathcal{W}_4(\mathcal{X}_h) \\ [x,\ z,\ w] &\longmapsto \left[\frac{z}{x}w,\ 2F_3\Big(1,\ \frac{z}{x}\Big)x + F_2\Big(1,\ \frac{z}{x}\Big)w,\ w\right]\,. \end{aligned}$$

The defining polynomial of $\mathcal{W}_5(\mathcal{X}_h)$ is

$$\begin{aligned} f_5(x,\ y,\ w) &:= f_4\Big(\frac{x}{A^2t^3},\ y,\ w\Big) \\ &= y^2 w - x^3 - q^2 x^2 w - q(pq-r)xw^2 + \{p^3 - (pq-r)^2\}w^3\,. \end{aligned}$$

Then we have an isomorphism

$$\begin{aligned} \mathcal{W}_4(\mathcal{X}_h) &\longrightarrow \mathcal{W}_5(\mathcal{X}_h) \\ [x,\ y,\ w] &\longmapsto [A^2t^3x,\ y,\ w]\,. \end{aligned}$$

There is a change of variables preserving the Weierstrass form defined by

$$\begin{aligned}\mathcal{W}_5(\mathfrak{X}_h) &\longrightarrow \mathcal{W}(\mathfrak{X}_h)\\ [x,\ y,\ w] &\longmapsto \left[\frac{1}{u^2}(x-pw),\ \frac{1}{u^3}\{y-qx+(pq-r)w\},\ w\right],\end{aligned}$$

where $u = 1/b$. □

LEMMA 2.17. *With notation as in the proof of Proposition 2.15, if* $\mathrm{char}\,(F) = 3$, $c \neq 0$, *and the point* $P_1 \in \vartheta_0(\mathfrak{X}_h)$ *is an inflexion point then there exists an isomorphism* $\vartheta : \mathfrak{X}_h \simeq \mathcal{W}(\mathfrak{X}_h)$ *defined over* $F(t)$.

Proof. The condition that the point P_1 is an inflexion point implies $A = 0$, and $B \neq 1$ by the irreducibility of $\mathfrak{X}_h$. Here we give only isomorphisms

$$\mathcal{W}(\mathfrak{X}_h) \longrightarrow \mathcal{W}_6(\mathfrak{X}_h) \longrightarrow \vartheta_0(\mathfrak{X}_h)\,.$$

The defining polynomial of $\mathcal{W}_6(\mathfrak{X}_h)$ is

$$f_6(x,\ y,\ z) := f_0\left(-x,\ (B-1)tz,\ \frac{y}{(B-1)t}\right).$$

Then we have an isomorphism

$$\begin{aligned}\mathcal{W}_6(\mathfrak{X}_h) &\longrightarrow \vartheta_0(\mathfrak{X}_h)\\ [x,\ y,\ z] &\longmapsto \left[-x,\ (B-1)tz,\ \frac{y}{(B-1)t}\right].\end{aligned}$$

There is a change of variables preserving the Weierstrass form defined by

$$\begin{aligned}\mathcal{W}(\mathfrak{X}_h) &\longrightarrow \mathcal{W}_6(\mathfrak{X}_h)\\ [x,\ y,\ z] &\longmapsto [u^2x+pz,\ u^3y+u^2qx+rz,\ z]\,,\end{aligned}$$

where

$$\begin{aligned}u &= \frac{1}{c}\,,\\ p &= \{B^2+B(t+1)+1\}t\,,\\ q &= (B-1)t\,,\\ r &= (-B^3+B^2t+Bt+1)t^2\,.\end{aligned}$$

□

2.3. The proof of (III)

LEMMA 2.18. *Let E be an elliptic curve defined over an arbitrary field L with an L-rational* 3*-torsion point. Then E has the Weierstrass equation*

$$y^2+Axy+By = x^3 \quad (A,\ B \in L)\,.$$

Proof. Since E is an elliptic curve defined over L, the defining equation of E as a plane curve is given by the Weierstrass equation

$$y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6 \quad (a_1,\ a_2,\ a_3,\ a_4,\ a_6 \in L)\,.$$

Making a linear transformation which sends the L-rational 3-torsion point on E to the origin $(0,\ 0)$, we obtain the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x$$

with the 3-torsion point $(0,\ 0)$. By calculations of derivatives, it is easy to see that the point $(0,\ 0)$ is smooth if and only if $a_3 \neq 0$ or $a_4 \neq 0$. If $a_3 = 0$ and $a_4 \neq 0$ then the point $(0,\ 0)$ is of order 2. Thus a_3 must be non-zero. Making the substitution

$$x \to x\,, \quad y \to y + \frac{a_4}{a_3}x$$

yields the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2\,.$$

Since the tangent line to the curve at $(0,\ 0)$ has slope 0, the condition that the point $(0,\ 0)$ is of order 3 implies $a_2 = 0$. We thus have the Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3$$

as stated in the lemma. □

The statement (III) can be proven as follows. Let E be any elliptic curve defined over $F(t)$ representing a class in $\mathcal{E}_{K_t/F(t)}$. By the property (i) and Lemma 2.18, the elliptic curve E has a Weierstrass equation

$$y^2 + Axy + By = x^3 \quad (A,\ B \in F(t))\,.$$

Then the associated 3-isogeny $\phi : E \to \widehat{E} = E/\langle(0,\ 0)\rangle$ is given by ([5])

$$(x,\ y) \longmapsto \left(\frac{x^3 + B(Ax + B)}{x^2},\ \frac{x^3(y + 4B) - B(Ax + B)^2 + By(y - B)}{x^3}\right)\,,$$

where the elliptic curve $\widehat{E}$ is defined by the equation

$$y^2 + Axy - 9By = x^3 - (A^3 + 27B)B\,.$$

The property (ii) implies that there exists some point $P = (k,\ l) \in \widehat{E}(F(t))$ such that the roots of the equations

$$\frac{x^3 + B(Ax + B)}{x^2} = k\,, \quad \frac{x^3(y + 4B) - B(Ax + B)^2 + By(y - B)}{x^3} = l \tag{2.1}$$

generate the cyclic cubic extension K_t over $F(t)$. Let $\alpha \in K_t$ be one of the roots of the equation on the left-hand side of (2.1). Note that this equation has discriminant $B^2(Ak - 9B + 2l)^2 \in F(t)^{*2}$, and every root of the equation on the right-hand side of (2.1) is contained in the field $F(t)(\alpha) = K_t$. Then one can verify that

$$\frac{2A^2B + k(-3B + l)}{Ak - 9B + 2l} + \frac{A^3B + 27B^2 - 10Bl + l^2 + Ak(-4B + l)}{B(Ak - 9B + 2l)}\alpha + \frac{3AB - k^2}{B(Ak - 9B + 2l)}\alpha^2$$

is another root of the equation on the left-hand side of (2.1). We may choose a generator σ of the Galois group $\mathrm{Gal}(K_t/F(t))$ such that α^σ is the above root, and adopt $\{1,\ \alpha,\ \alpha^2\}$ as a basis of $K_t/F(t)$ for the norm surface $\mathcal{X}$. It is sufficient to prove that there exists some $h = [a,\ b,\ c,\ -a] \in \check{\mathbb{P}}^3(F(t))$ satisfying $\mathrm{Tr}(\gamma_h) = A$ and $\gamma_h = -B/\alpha^{\sigma^2}$. If this is the case, then $\mathrm{N}(\gamma_h) = B$ by the equation on the left-hand side of (2.1), and hence, by the result of (II), the cubic curve $\mathcal{X}_h$ for the plane $h = [a, b, c, -a] \in \check{\mathbb{P}}^3(F(t))$ has the Weierstrass equation

$$y^2 + Axy + By = x^3 ,$$

which is exactly the defining equation of E as has been chosen above. Let

$$a = \frac{A}{B(Ak - 9B + 2l)} , \quad b = -\frac{3}{Ak - 9B + 2l} , \quad c = -\frac{k}{Ak - 9B + 2l} .$$

Then one can verify the equalities $\mathrm{Tr}(\gamma_h) = A$ and $\gamma_h = -B/\alpha^{\sigma^2}$ using the defining equation of $\widehat{E}$. Therefore we have $\{E/F(t)\} \in \mathcal{E}(\mathcal{X})$. This completes the proof of (III).

REMARK 2.19. The set $\mathcal{E}(\mathcal{X})$ does not depend on the choice of a basis of $K_t/F(t)$.

REMARK 2.20. We can also consider the following conditions instead of the condition $\mathrm{Tr}(\gamma_h) = A,\ \gamma_h = -B/\alpha^{\sigma^2}$. For the condition $\mathrm{Tr}(\gamma_h) = A,\ \gamma_h = -B/\alpha$, we can choose the constants

$$a = \frac{A}{B(Ak - 9B + 2l)} , \quad b = \frac{Ak - 3B + l}{B(Ak - 9B + 2l)} , \quad c = \frac{-A^2B + Ak^2 - 4Bk + kl}{B(Ak - 9B + 2l)} .$$

For the condition $\mathrm{Tr}(\gamma_h) = A,\ \gamma_h = -B/\alpha^{\alpha}$, we can choose the constants

$$a = \frac{A}{B(Ak - 9B + 2l)} , \quad b = \frac{6B - l}{B(Ak - 9B + 2l)} , \quad c = \frac{-A^2B + 5Bk - kl}{B(Ak - 9B + 2l)} .$$

We end this subsection with the following characterization of elliptic curves with a rational 3-torsion point.

THEOREM 2.21. *Let E be any elliptic curve given by the Weierstrass equation*

$$y^2 + Axy + By = x^3 \quad (A,\ B \in F) ,$$

and let ϕ be a 3-isogeny $E \to \widehat{E} = E/\langle(0, 0)\rangle$. Then for any cyclic cubic extension K/F the following two conditions are equivalent.

(ii) *There exists some point $P \in \widehat{E}(F)$ such that $K = F(\phi^{-1}(P))$.*
(iv) *There exists some element $\gamma \in K^*$ satisfying* $\mathrm{Tr}(\gamma) = A$ *and* $\mathrm{N}(\gamma) = B$.

Proof. If (ii) holds then, as we have seen in the proof of (III), we can find an element $\gamma \in K^*$ satisfying $\mathrm{Tr}(\gamma) = A$ and $\mathrm{N}(\gamma) = B$. Thus, we assume that (iv) $\mathrm{Tr}(\gamma) = A$ and $\mathrm{N}(\gamma) = B$ for some $\gamma \in K^*$. In order to prove (ii), by the results of Theorem 1.2 and Theorem 2.5, it suffices to show the existence of a plane $h = [a, b, c, -a] \in \check{\mathbb{P}}^3(F)$ satisfying $\gamma_h = \gamma$ with the basis $\{1, \beta_1, \beta_2\}$ of K/F. Here $\beta_1 = \alpha$, $\beta_2 = 1/(1-\alpha)$ with a root α of the equation $g(x; t) = 0$. Since $\{1, \alpha, \alpha^2\}$ is also a basis of K/F, there is a linear combination $\gamma = k + l\alpha + m\alpha^2$ for some $k, l, m \in F$. Recall the proof of Lemma 2.2. Then the element

$$\gamma_h = \begin{vmatrix} 1 & \beta_1 & \beta_2 \\ 1 & \beta_1^\sigma & \beta_2^\sigma \\ a & b & c \end{vmatrix} = \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} t-3 & t+4 & 2 \\ -t^2 - t & -2t - 1 & -t - 2 \\ -t & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

coincides with the element

$$\gamma = \begin{pmatrix} 1 & \alpha & \alpha^2 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix}$$

if and only if

$$\begin{pmatrix} t-3 & t+4 & 2 \\ -t^2-t & -2t-1 & -t-2 \\ -t & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}.$$

The coefficient matrix on the left-hand side has determinant $\delta(g)$, which is non-zero, and hence for any element $\gamma \in K^*$ there exists a plane $h = [a, b, c, -a] \in \check{\mathbb{P}}^3(F)$ satisfying $\gamma_h = \gamma$. □

REMARK 2.22. By Theorem 1.2, Theorem 2.5 and Theorem 2.21, the set $\mathcal{E}(\mathcal{X})$ consists of all F-isomorphism classes of elliptic curves

$$y^2 + \mathrm{Tr}(\gamma)xy + \mathrm{N}(\gamma)y = x^3$$

varying the element γ through K^*.

2.4. The proof of (IV)

In order to prove (IV), we need the following key lemma. This lemma gives a relation between the constants γ_h and $\gamma_{h'}$ via the action $\tilde{\rho}(P)(h) = h'$ on planes in $\check{\mathbb{P}}^3$. Let L denote an arbitrary field containing $F(t)$.

LEMMA 2.23. *For any $h = [a, b, c, d] \in \check{\mathbb{P}}^3(L)$ and any $P = [x, y, z, w] \in \mathcal{X}_{w\neq 0}(L)$, let*

$$[a', b', c', d'] := [a, b, c, d] \cdot {}^t\tilde{\rho}(P)^{-1} (= \tilde{\rho}(P)(h)).$$

Then

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^\sigma & \alpha_2^\sigma \\ a & b & c \end{vmatrix} d' = \left(\frac{x}{w} + \frac{y}{w}\alpha_1^{\sigma^2} + \frac{z}{w}\alpha_2^{\sigma^2}\right) \begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^\sigma & \alpha_2^\sigma \\ a' & b' & c' \end{vmatrix} d.$$

Proof. We first recall the notation in Lemma 2.2. Then the statement of lemma is

$$|M(a, b, c)|d' = \left\{\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right) \cdot {}^t(1, \alpha_1, \alpha_2)^{\sigma^2}\right\} |M(a', b', c')|d.$$

This is equivalent to

$$|M(a, b, c) \cdot M_0|d' = \left\{\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right) {}^tM_0^{-1} \cdot (1, \beta_1, \beta_2)^{\sigma^2}\right\} |M(a', b', c') \cdot M_0|d,$$

and hence it suffices to show the lemma for the basis $\{1, \beta_1, \beta_2\}$ of $K_t/F(t)$. For the basis $\{1, \beta_1, \beta_2\}$, the matrix $\tilde{\rho}(P)$ can be expressed by

$$\tilde{\rho}(P) = \begin{pmatrix} \frac{x}{w} & \frac{y}{w} & \frac{z}{w} & 0 \\ 2\frac{y}{w} - \frac{z}{w} & \frac{x}{w} - (t+1)\frac{y}{w} & -\frac{y}{w} + \frac{z}{w} & 0 \\ -\frac{y}{w} + (t+2)\frac{z}{w} & \frac{z}{w} & \frac{x}{w} + \frac{y}{w} - t\frac{z}{w} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(L). \tag{2.2}$$

We thus have

$$[a, b, c, d] = [a', b', c', d'] \cdot {}^t\tilde{\rho}(P)$$

$$= {}^t\begin{bmatrix} a'x + b'y + c'z \\ a'(2y - z) + b'\{x - (t+1)y\} + c'(-y + z) \\ a'\{-y + (t+2)z\} + b'z + c'(x + y - tz) \\ d'w \end{bmatrix} \in \check{\mathbb{P}}^3(L)\,.$$

Since $\beta_1 = \alpha$, $\beta_2 = 1/(1-\alpha)\ (= \beta_1^\sigma)$ is the roots of the cubic equation $g(x; t) = 0$, a direct calculation shows that

$$\begin{vmatrix} 1 & \beta_1 & \beta_2 \\ 1 & \beta_1^\sigma & \beta_2^\sigma \\ a & b & c \end{vmatrix} d'w = (x + y\beta_1^{\sigma^2} + z\beta_2^{\sigma^2}) \begin{vmatrix} 1 & \beta_1 & \beta_2 \\ 1 & \beta_1^\sigma & \beta_2^\sigma \\ a' & b' & c' \end{vmatrix} d\,.$$

□

In the following argument, we consider γ_h varying planes $\{h\}$ through $\check{\mathbb{P}}^3(L)$. For simplicity, we use the notation

$$\mathrm{Tr}(a, b, c) := \mathrm{Tr}(\gamma_h)\,, \quad \mathrm{N}(a, b, c) := \mathrm{N}(\gamma_h)\,,$$

and thus we regard them as polynomials in a, b, c. For any $h = [a, b, c, d] \in \check{\mathbb{P}}^3(L)$, let $\mathcal{C}_h$ be a projective curve in $\mathbb{P}^3$ defined by the homogeneous equations

$$\mathrm{Tr}(x, y, z) = \mathrm{Tr}(a, b, c)w\,, \quad \mathrm{N}(x, y, z) = \mathrm{N}(a, b, c)w^3\,, \quad ([x, y, z, w] \in \mathbb{P}^3)\,.$$

Clearly, the curve $\mathcal{C}_h$ is well-defined up to isomorphisms, which does not depend on the choice of the representative (a, b, c) in $[a, b, c, d] \in \check{\mathbb{P}}^3(L)$. Then using Lemma 2.23, we have the following identification.

PROPOSITION 2.24. *Let $h = [a, b, c, d] \in \check{\mathbb{P}}^3(L)$ be a plane passing through the point $\mathcal{O}$. Assume that $\mathrm{N}(a, b, c)$ is non-zero. Then there is an isomorphism $\xi : \mathcal{X}_h \xrightarrow{\sim} \mathcal{C}_h$ over $F(t, a, b, c)$ given by the rational map $P \mapsto [a, b, c, 1] \cdot {}^t\tilde{\rho}(P) (= \tilde{\rho}(P^{-1})([a, b, c, 1]))$.*

Proof. If $P = [x, y, z, w]$ is a point on $\mathcal{X}_{h, w\neq 0}$, then it follows from Lemma 2.23 that $[a', b', c', u] := [a, b, c, 1] \cdot {}^t\tilde{\rho}(P)$ satisfies

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^\sigma & \alpha_2^\sigma \\ a' & b' & c' \end{vmatrix} = \left(\frac{x}{w} + \frac{y}{w}\alpha_1^{\sigma^2} + \frac{z}{w}\alpha_2^{\sigma^2}\right) \begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^\sigma & \alpha_2^\sigma \\ a & b & c \end{vmatrix} u\,. \tag{2.3}$$

Since Lemma 2.23 is valid for any basis of $K_t/F(t)$, the equation (2.3) holds even if we make the substitution $\alpha_i \mapsto \alpha_i^\sigma$ or $\alpha_i \mapsto \alpha_i^{\sigma^2}$. Taking norm N of the equation (2.3) yields

$$\mathrm{N}(a', b', c') = \mathrm{N}(a, b, c)u^3\,,$$

since the point $P = [x,\ y,\ z,\ w]$ lies on the norm surface $\mathcal{X}$. On the one hand, taking trace Tr of the equation (2.3), one can also easily verify

$$\mathrm{Tr}(a',\ b',\ c') = \left(a\frac{x}{w} + b\frac{y}{w} + c\frac{z}{w}\right)\begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^{\sigma} & \alpha_2^{\sigma} \\ 1 & \alpha_1^{\sigma^2} & \alpha_2^{\sigma^2} \end{vmatrix} u\,.$$

This $\mathrm{Tr}(a',\ b',\ c')$ coincides with (Remark 2.9)

$$\mathrm{Tr}(a,\ b,\ c)u = a\begin{vmatrix} 1 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1^{\sigma} & \alpha_2^{\sigma} \\ 1 & \alpha_1^{\sigma^2} & \alpha_2^{\sigma^2} \end{vmatrix} u$$

if and only if the equation $a(x - w) + by + cz = 0$ holds. From the assumption $\mathcal{O} \in h$, the equation $a(x - w) + by + cz = 0$ holds, and hence

$$\mathrm{Tr}(a',\ b',\ c') = \mathrm{Tr}(a,\ b,\ c)u\,.$$

Thus, we have the non-constant rational map

$$\begin{aligned} \xi : \mathcal{X}_h &\longrightarrow \mathcal{C}_h \\ P &\longmapsto [a,\ b,\ c,\ 1] \cdot {}^t\tilde{\rho}(P)\,, \end{aligned}$$

whose restriction to $\mathcal{X}_{h,w\neq 0}$ is regular. Now we consider the following commutative diagram.

$$\begin{tikzcd} \mathcal{X}_h \arrow[r, "\xi"] & \mathcal{C}_h \\ \mathcal{X}'_{h'} \arrow[u, "\iota"] \arrow[ur, "\xi\circ\iota"'] & \end{tikzcd}$$

Here $\mathcal{X}'_{h'}$ denotes another model of the elliptic curve $\mathcal{X}_h$ whose equation is defined using the norm surface $\mathcal{X}$ with the basis $\{1,\ \beta_1,\ \beta_2\}$ of $K_t/F(t)$, and ι denotes a canonical isomorphism between $\mathcal{X}_h$ and $\mathcal{X}'_{h'}$ induced by some matrix M_0 as in Lemma 2.2. For any point $[a',\ b',\ c',\ u] \in \mathcal{C}_h$, again using the expression (2.2), we find the linear equation

$$\begin{aligned} [a',\ b',\ c',\ u] &= \xi \circ \iota([x,\ y,\ z,\ w]) \\ &= [a,\ b,\ c,\ 1] \cdot \begin{pmatrix} \frac{x}{w} & 2\frac{y}{w} - \frac{z}{w} & -\frac{y}{w} + (t+2)\frac{z}{w} & 0 \\ \frac{y}{w} & \frac{x}{w} - (t+1)\frac{y}{w} & \frac{z}{w} & 0 \\ \frac{z}{w} & -\frac{y}{w} + \frac{z}{w} & \frac{x}{w} + \frac{y}{w} - t\frac{z}{w} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= [x,\ y,\ z,\ w] \cdot \begin{pmatrix} a & b & c & 0 \\ b & 2a - b(t+1) - c & -a + c & 0 \\ c & -a + c & a(t+2) + b - ct & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The determinant of the latter 4×4-matrix is

$$-\frac{\mathrm{N}(a,\,b,\,c)}{\delta(g)} = -b^3+c^3+b^2ct+bc^2(t+3)+a^3(2t+3)-a^2\{b(t+3)+3c\}t$$
$$-a\{b^2(2t+3)-bc(t^2+t-3)-c^2(t-3)\},$$

which is non-zero by the assumption $\mathrm{N}(a,\,b,\,c) \neq 0$. This implies that the rational map $\xi \circ \iota$ is an isomorphism defined over $F(t,\,a,\,b,\,c)$, and hence the map ξ is an isomorphism. □

REMARK 2.25. For example, the assumption $(\mathrm{N}(\gamma_h) =)\mathrm{N}(a,\,b,\,c) \neq 0$ in Proposition 2.24 holds under one of the following conditions:

- $\mathcal{X}_h$ is smooth (*i.e.* an elliptic curve) (by Theorem 2.5 and Remark 2.7).
- The field K_t is not contained in the ground field L (by Lemma 2.2).

REMARK 2.26. In the proof of Proposition 2.24, the unique solution of the linear equation $[a',\,b',\,c',\,u] = \xi \circ \iota([x,\,y,\,z,\,w])$ is

$$x = -\frac{\delta(g)}{\mathrm{N}(a,\,b,\,c)}\Big\{(a^2(2t+3)-b^2(t+1)+c^2(t-1)+bc(t^2+t-1)-a(b(t+3)+3c))ta'$$
$$+(-b^2+c^2+bct-a(b(t+2)+c))b'+(-a(b+2c)+c(b(t+2)+c))c'\Big\},$$

$$y = -\frac{\delta(g)}{\mathrm{N}(a,\,b,\,c)}\Big\{(-b^2+c^2+bct-a(b(t+2)+c))a'+(a^2(t+2)+a(b-tc)-c^2)b'$$
$$+(a^2-ac+bc)c'\Big\},$$

$$z = -\frac{\delta(g)}{\mathrm{N}(a,\,b,\,c)}\Big\{(-a(b+2c)+c(b(t+2)+c))a'+(a^2-ac+bc)b'$$
$$+(2a^2-b^2-a(b(t+1)+c))c'\Big\},$$

$$w = u.$$

This defines the inverse map for $\xi \circ \iota$.

2.5. The proof of (V)

In this subsection, let L denote an arbitrary field containing $F(t)$ but not $\sqrt{-3}$. At the end of this subsection, we will give a proof of Theorem 1.4 as a consequence of the results (IV) and (V). We first state a characterization of the isomorphic elliptic curves $\{\mathcal{X}_h\}$ for $h \in \check{\mathbb{P}}^3(L)$ passing through the point $\mathcal{O}$.

THEOREM 2.27. *Let $h \in \check{\mathbb{P}}^3(L)$ be a plane passing through the point $\mathcal{O}$ such that the section $\mathcal{X}_h$ is an elliptic curve. We also let*

$$\mathcal{H}_h(L) := \left\{h' \in \check{\mathbb{P}}^3(L) \;\middle|\; \mathcal{O} \in h' \text{ and } \mathcal{X}_{h'} \text{ is isomorphic to } \mathcal{X}_h \text{ over } L\right\}.$$

Then there exists a bijection

$$\begin{aligned}\mathcal{X}_{h,w\neq 0}(L) &\xrightarrow{\sim} \mathcal{H}_h(L)\\ P &\longmapsto \tilde{\rho}(P^{-1})(h)\,.\end{aligned}$$

In particular, for any planes $h,\,h' \in \check{\mathbb{P}}^3(L)$ such that $\mathcal{X}_h,\,\mathcal{X}_{h'}$ are elliptic curves with the point $\mathcal{O}$, if $\mathcal{X}_h$ is isomorphic to $\mathcal{X}_{h'}$ over L then there exists some point $P \in \mathcal{X}_{h,w\neq 0}(L)$

such that $\mathcal{X}_h = P \cdot \mathcal{X}_{h'}$, *where* $\cdot$ *stands for the linear algebraic group action as introduced in* §1.

Proof. Combining Proposition 2.24 and Proposition 2.28 readily yields the bijection $\eta \circ \xi : \mathcal{X}_{h,w\neq 0}(L) \xrightarrow{\sim} \mathcal{H}_h(L)$ given by the map $P \mapsto \tilde{\rho}(P^{-1})(h)$. □

Theorem 2.27 says that the rational points on the affine curve $\mathcal{X}_{h,w\neq 0}$ parametrize the set $\mathcal{H}_h$; that is, the isomorphic elliptic curves $\{\mathcal{X}_h\}$ passing through the point $\mathcal{O}$.

PROPOSITION 2.28. *Let* $h = [a, b, c, d] \in \check{\mathbb{P}}^3(L)$ *be a plane passing through the point* $\mathcal{O}$. *Assume that* $\mathcal{X}_h$ *is smooth* (*i.e. an elliptic curve*). *Then there is a bijection* $\eta : \mathcal{C}_{h,w\neq 0}(L) \xrightarrow{\sim} \mathcal{H}_h(L)$ *given by the map* $[a', b', c', 1] \mapsto [a', b', c', -a']$.

Proof. By Theorem 2.5, we can define the map

$$\begin{aligned} \eta : \mathcal{C}_{h,w\neq 0}(L) &\longrightarrow \mathcal{H}_h(L) \\ [a', b', c', 1] &\longmapsto [a', b', c', -a'] \,. \end{aligned}$$

We first show that the map η is injective. If $\eta([a', b', c', 1]) = \eta([a'', b'', c'', 1])$ then there exists some $u \in L^*$ such that $[a', b', c', u] = [a'', b'', c'', 1] \in \mathcal{C}_{h,w\neq 0}$, and hence

$$\mathrm{Tr}(a, b, c) = \mathrm{Tr}(a, b, c)u, \quad \mathrm{N}(a, b, c) = \mathrm{N}(a, b, c)u^3 \,.$$

Since $\mathrm{N}(a, b, c) \neq 0$ from the assumption that $\mathcal{X}_h$ is an elliptic curve (Remark 2.7), we have $u^3 = 1$. Hence $u = 1$ by the condition $\sqrt{-3} \notin L$ (There is a possibility that $\mathrm{Tr}(a, b, c) = 0$). This proves the injectivity of η.

We next show that the map η is surjective. Let $h' = [a', b', c', -a']$ be any plane in $\mathcal{H}_h$. Again using Theorem 2.5 yields isomorphisms

$$\begin{aligned} \mathcal{X}_h &\xrightarrow{\sim} \mathcal{W}(\mathcal{X}_h) : y^2z + \mathrm{Tr}(a, b, c)xyz + \mathrm{N}(a, b, c)yz^2 = x^3, \\ \mathcal{X}_{h'} &\xrightarrow{\sim} \mathcal{W}(\mathcal{X}_{h'}) : y^2z + \mathrm{Tr}(a', b', c')xyz + \mathrm{N}(a', b', c')yz^2 = x^3. \end{aligned}$$

Each of the two elliptic curves $\mathcal{W}(\mathcal{X}_h)$, $\mathcal{W}(\mathcal{X}_{h'})$ has a 3-torsion subgroup generated by the point $[0, 0, 1] \in \mathbb{P}^3$. From the property of $\mathcal{H}_h$, there exists an isomorphism

$$\iota : \mathcal{W}(\mathcal{X}_h) \xrightarrow{\sim} \mathcal{X}_h \xrightarrow{\sim} \mathcal{X}_{h'} \xrightarrow{\sim} \mathcal{W}(\mathcal{X}_{h'})$$

over L fixing the point $[0, 1, 0] \in \mathbb{P}^3$ at infinity (If necessary, make a translation). Since the map ι fixes $[0, 1, 0]$ and preserves the Weierstrass form, we have

$$\begin{aligned} \iota : \mathcal{W}(\mathcal{X}_h) &\longrightarrow \mathcal{W}(\mathcal{X}_{h'}) \\ [x, y, z] &\longmapsto \left[u^2x + pz, u^3y + u^2qx + rz, z\right] \end{aligned}$$

for some $u, p, q, r \in L$, $u \neq 0$ (III-§1 in [7]). Since the point $[0, 0, 1] \in \mathcal{W}(\mathcal{X}_h)$ is a 3-torsion point, the point $\iota([0, 0, 1]) = [p, r, 1]$ must be a 3-torsion point on $\mathcal{W}(\mathcal{X}_{h'})$. Now we prove $[p, r, 1] = \pm[0, 0, 1]$ dividing into two cases where $\mathrm{char}\,(F) = 3$ or not.

We first assume that $\mathrm{char}\,(F) = 3$. Then it is easily seen from the 3-division polynomial for $\mathcal{W}(\mathcal{X}_h)$ that the full 3-torsion subgroup of $\mathcal{W}(\mathcal{X}_{h'})$ is generated by the point $[0, 0, 1]$ (see also III-§6 in [7]), and hence $[p, r, 1] = \pm[0, 0, 1]$. We next assume that $\mathrm{char}\,(F) \neq 3$. Then, by the condition $\sqrt{-3} \notin L$ we have $[p, r, 1] = \pm[0, 0, 1]$, because if this is not the case then the well-known property of Weil pairings asserts that the field L contains a primitive cube root of unity (III-§8 in [7]).

If necessary, substituting $[-1] \circ \iota$ for the map ι, we have

$$\iota([0,\ 0,\ 1]) = [p,\ r,\ 1] = [0,\ 0,\ 1]\,.$$

Here $[-1]$ stands for the multiplication by -1 map on $\mathcal{W}(\mathcal{X}_{h'})$, which is an automorphism. This implies that $p = q = r = 0$ (III-§1 in [7]), and thus

$$\mathrm{Tr}(a',\ b',\ c') = \mathrm{Tr}(a,\ b,\ c)u, \quad \mathrm{N}(a',\ b',\ c') = \mathrm{N}(a,\ b,\ c)u^3\,.$$

Therefore $[a',\ b',\ c',\ u] \in \mathcal{C}_h(L)$ and $\eta([a',\ b',\ c',\ u]) = [a',\ b',\ c',\ -a']$. The surjectivity of η now follows. $\square$

Finally we prove Theorem 1.4. Let $\mathcal{X}_{h_1}$, $\mathcal{X}_{h_2}$ be elliptic curves for planes $h_1,\ h_2 \in \check{\mathbb{P}}^3(F)$, respectively. Since $\mathcal{X}_{h_i}(F) = \mathcal{X}_{h_i, w\neq 0}(F)$ is a non-empty subset of $\mathcal{X}_{w\neq 0}(F)$, there exist some points $Q_i \in \mathcal{X}_{h_i}(F)$ so that the point $\mathcal{O}$ lies on the planes $h'_i = \tilde{\rho}(Q_i^{-1})(h_i)$ and $\mathcal{X}_{h_i}$ is isomorphic to $\mathcal{X}_{h'_i}$ over F. By Theorem 2.27, $\mathcal{X}_{h_1}$ is isomorphic to $\mathcal{X}_{h_2}$ over L if and only if there is some point $R \in \mathcal{X}_{h'_1, w\neq 0}(L)$ such that $\mathcal{X}_{h'_1} = R \cdot \mathcal{X}_{h'_2}$, namely, $\mathcal{X}_{h_1} = P \cdot \mathcal{X}_{h_2}$ for $P := Q_1 \cdot R \cdot Q_2^{-1} \in \mathcal{X}_{w\neq 0}(L)$. This completes the proof.

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